ELSEVIER

Available at
www.ComputerScienceWeb.com
POWERED BY SCIENCE DIRECT®

Theoretical Computer Science 305 (2003) 135–158

Theoretical
Computer Science

www.elsevier.com/locate/tcs

A non-Hausdorff quaternion multiplication

K.A. Hardie[a,*], S. Salbany[b], J.J.C. Vermeulen[c,1], P.J. Witbooi[d]

[a] *Department of Mathematical Technology, Technikon Pretoria, Private Bag X680, 0001 Pretoria, South Africa*

[b] *Department of Mathematics, Applied Mathematics and Astronomy, University of South Africa, P.O. Box 392, 0001 Pretoria, South Africa*

[c] *Department of Mathematics and Applied Mathematics, University of Cape Town, 7700 Rondebosch, South Africa*

[d] *Department of Mathematics, University of the Western Cape, Private Bag X17, 7535 Bellville, South Africa*

Abstract

We denote by $(\mathbb{S}^3)'$ the barycentric subdivision of the minimal model $\mathbb{S}^3$ of the three-dimensional sphere in the category of finite posets and order-preserving functions, $\mathrm{op}(X)$ is the poset obtained by reversing the order relations in a poset X. We describe a finite model of a quaternion multiplication in the form of a morphism $\mathrm{op}(\mathbb{S}^3)' \times (\mathbb{S}^3)' \to \mathbb{S}^3$ that restricts to weak homotopy equivalences on the axes. For such multiplications a version of *Hopf's construction* can be defined that yields finite models of non-trivial homotopy classes.

PACS: 55Q25; 55Q40; 06A07; 52B99; 20G20

Keywords: Quaternions; Posets; Simplicial complexes; Barycentric subdivision; Hopf map; Hopf construction

0. Introduction

Elements of homotopy groups of spaces have been used to characterise physical phenomena. For example, as expounded by Nakahara [11], *line, point* and *ring defects* in nematic liquid crystals are associated with the generators of $\pi_i(\mathbb{R}P^2)$ $(i = 1, 2)$, the Hopf classes $\eta_2 \in \pi_3(S^2)$ and $\nu_4 \in \pi_7(S^4)$ with the *magnetic monopole*, respectively

* Department of Mathematical Technology, Technikon Pretoria, Private Bag X680, 0001 Pretoria, South Africa.

E-mail addresses: hardieka@iafrica.com (K.A. Hardie), salbasdo@alpha.unisa.ac.za (S. Salbany), pwitbooi@uwc.ac.za (P.J. Witbooi).

[1] The author died in a tragic accident on 2001.02.11.

PII: S0304-3975(02)00703-X

the *instanton* of unit strength. Representative maps for many such classes are well understood in terms of formulae and equations and in terms of the bundle structures they define.

To understand the geometry of representative maps it would be helpful to be able to express the maps as (continuous) functions in *finite* terms. One way to proceed would be to represent them as simplicial maps between finite simplicial complexes. However there is some difficulty to do so even for a given map. The *simplicial approximation* theorem, at least in its standard form derived from the Lebesgue Number theory of an open covering of a compact space [9, Theorem 1.8.1], is non-constructive. Although computation of simplicial approximations is possible from geometric properties of the map, that is an ox behind a wagon. What seems to be needed is a construction of the map itself in finite terms as it arises. To this end we propose to work in an appropriate category of finite spaces, specifically the category of finite T_0-spaces. However, there is a well-known isomorphism between this category and the category of finite partially ordered sets (posets) and order preserving functions. (Set $x \leqslant y$ if and only if $y \in \overline{(x)}$, in the reverse direction $\{x \mid x \leqslant y\}$ is the smallest open set containing y.)

Classically, new elements of the homotopy groups of spheres arose by the application of Hopf's *construction* to the maps of products associated with the operation in the topological groups S^1 and S^3 [5,6]. Later, this constructional method was extended to handle maps of products associated with H-spaces (e.g. S^7) and more generally with vanishing Whitehead products. We point out in Section 3 that if such a map of products exists then it can *always* be realised (in some form) at the finite poset level but the argument is not constructive: the finite version has to be sought for and discovered by means of a search process. Indeed this is consistent with the general situation in homotopy theory. A large system of algebraic machinery has evolved asserting that maps of a certain type do not exist. Where no counterindication is available it is always a problem to find the map (or prove a stronger result).

Topology in the context of partial order is well established [2] and deep [13]. A connection between the homotopy theory of finite T_0-spaces and of compact polyhedra was established by Alexandroff [1] followed by McCord [10]. With each finite T_0-space is associated a finite simplicial complex $\mathscr{K}X$. The vertices of $\mathscr{K}X$ are the points of X. A subset V of X spans a simplex of $\mathscr{K}X$ if and only if every two point subset of V is connected in X. Conversely, if K is a finite simplicial complex, McCord defines a finite poset $\mathscr{X}K$ (the set of simplexes of K, with $\sigma \leqslant \tau$ if and only if σ is a face of τ). Then $\mathscr{K}$ and $\mathscr{X}$ are functorial and satisfy $\mathscr{K}\mathscr{X}K = K'$, the barycentric subdivision of K, [9]. He then defines a natural transformation

$$q : |\mathscr{K} - | \to 1$$

($|K|$ denotes the underlying polyhedron of K) as follows. If $x \in |\mathscr{K}X|$, let σ_x denote the unique open simplex of $\mathscr{K}X$ containing x. If $\sigma_x = (x_0, x_1, \cdots, x_r)$, where $x_0 < x_1 < \cdots < x_r$, he sets $qX(x) = x_0$ and proves the following theorem.

Theorem 0.1 (McCord). *For each finite* T_0 *space* X, $qX : |\mathscr{K}X| \to X$ *is a weak homotopy equivalence.*

The reader unfamiliar with the notion of weak homotopy equivalence might find it convenient to consult [16, p. 181]. For others the following brief reminder may suffice. An (ordinary) homotopy equivalence $f:Z\to X$ is a map for which there exists a *homotopy inverse*, i.e. a map $g:X\to Z$ such that the composite maps $fg:X\to X$ and $gf:Y\to Y$ are each homotopic to the respective identity maps $X\to X$ and $Y\to Y$. By functoriality of the *homotopy groups* $\pi_n(Z,z)$, every map $f:Z\to X$ induces a homomorphism $(n>0)$ and a function $(n=0)$

$$f_*:\pi_n(Z,z)\to\pi_n(X,f(z))\quad (n\geqslant 0,\ z\in Z).$$

If a function f as above has the property that all the functions f_* are bijections $(n\geqslant 0,\ z\in Z)$, then f is said to be a *weak (homotopy) equivalence*. It can be proved that every homotopy equivalence is a weak homotopy equivalence. According to a celebrated result of J.H.C. Whitehead, if f is a weak homotopy equivalence between CW-complexes (a class of spaces containing the polyhedra), then f is necessarily a homotopy equivalence.

We may regard the poset X as a *finite model* of $|\mathscr{K}X|$. One consequence of McCord's theorem is that there is no shortage of maps *into* X. However the situation is quite different when we look for maps *away from* X. To illustrate this point let us consider McCord's models of the n-spheres $(n=1,2,3)$.

$$\mathbb{S}^1=\begin{tikzcd} i & -i \\ 1 \arrow[u] \arrow[ur] & -1 \arrow[u] \arrow[ul] \end{tikzcd}\quad \mathbb{S}^2=\begin{tikzcd} j & -j \\ i \arrow[u] \arrow[ur] & -i \arrow[u] \arrow[ul] \\ 1 \arrow[u] \arrow[ur] & -1 \arrow[u] \arrow[ul] \end{tikzcd}\quad \mathbb{S}^3=\begin{tikzcd} k & -k \\ j \arrow[u] \arrow[ur] & -j \arrow[u] \arrow[ul] \\ i \arrow[u] \arrow[ur] & -i \arrow[u] \arrow[ul] \\ 1 \arrow[u] \arrow[ur] & -1 \arrow[u] \arrow[ul] \end{tikzcd}\tag{0.2}$$

First note (since no pair of points is separated by disjoint opens) that every map from one of these into a Hausdorff space is constant. Secondly, note that even self-maps are severely restricted: the only self-maps apart from the identities are *reflection* maps interchanging one or more pairs such as $(1,-1),(i,-i)$ etc., the respective *antipodal* maps that send each point into the corresponding point of opposite sign and *folding* maps of various kinds. In particular each self-map has degree $1,-1$ or 0. (The *degree* of a map is the degree of the simplicial map it induces.)

Representatives of more interesting homotopy classes are available if the domain poset is subdivided. For a poset X, its *barycentric subdivision* X' is the poset of chains of X. (A *chain* of X is a finite increasing sequence of points of X, the chains are ordered by inclusion.) The following consequence of the classical simplicial approximation theorem was given in [7]. It shows that each homotopy class in $\pi(|\mathscr{K}X|,|\mathscr{K}Y|)$ has a finite model.

Theorem 0.3 (Simplicial approximation) (Hardie-Vermeulen). *Let* $f:|\mathscr{K}X|\to|\mathscr{K}Y|$ *be a* (*continuous*) *map, where* X,Y *are finite* T_0. *Then there exists an integer* n *and a continuous map* $g:X^{(n)}\to Y$ *such that* $|\mathscr{K}g|\simeq f$.

Note, by way of example, that

$$
(\mathbb{S}^1)' = \begin{array}{ccc} -1+i & \longleftarrow i \longrightarrow & 1+i \\ \uparrow & & \uparrow \\ -1 & & 1 \\ \downarrow & & \downarrow \\ -1-i & \longleftarrow -i \longrightarrow & 1-i \end{array} = \begin{array}{ccc} \cdot & \longleftarrow i \longrightarrow & \cdot \\ \uparrow & & \uparrow \\ -1 & & 1 \\ \downarrow & & \downarrow \\ \cdot & \longleftarrow -i \longrightarrow & \cdot \end{array}
$$

and that a degree 2 map $(\mathbb{S}^1)' \to \mathbb{S}^1$ may be defined by winding the 8-point circle twice around $\mathbb{S}^1$.

The rather strange dearth of morphisms between finite posets provides us with additional motivation for our studies: when a map representative of a particular homotopy class first arises at the finite level it has comparatively few competitors. (Note, for example, that there is only *one* degree 2 map from $(\mathbb{S}^1)' \to \mathbb{S}^1$ that sends 1 to 1.) Pursuing minimal models has the effect of stripping irrelevant features of the relevant maps and their essential geometric features are economically exposed.

Three of the present authors have studied in [8] a finite model of the multiplication on the one-dimensional sphere and used it to construct a finite model of Hopf's class from the 3-sphere to the 2-sphere. There the model of the multiplication presented itself as an order-preserving function between posets

$$m : (\mathbb{S}^1)' \times (\mathbb{S}^1)' \to \mathbb{S}^1, \tag{0.4}$$

which, restricted to the *axes* $(\mathbb{S}^1)' \times (1)$ and $(1) \times (\mathbb{S}^1)'$, yielded weak homotopy equivalences.

It was natural to ask whether similar models were available for the multiplications on the 3-sphere and 7-sphere, and in particular, whether a single barycentric subdivision would again suffice. In the present paper we answer this question in the affirmative (for the 3-sphere) by describing an order-preserving function

$$v : \mathrm{op}(\mathbb{S}^3)' \times (\mathbb{S}^3)' \to \mathbb{S}^3. \tag{0.5}$$

(By a famous result due to J.F. Adams, $n = 1, 3, 7$ are the only values for which the n-sphere S^n admits a multiplication with a two-sided unit. For details, see [4, pp. 549–551].) In order to describe the function (0.5) we prefer to write the chain

$$x = (\varepsilon_1 1, \varepsilon_2 i, \varepsilon_3 j, \varepsilon_4 k) \in (\mathbb{S}^3)' \quad (\varepsilon_r \in \{-1, 0, 1\},\ 1 \leqslant r \leqslant 4)$$

in the sum form

$$x = \varepsilon_1 1 + \varepsilon_2 i + \varepsilon_3 j + \varepsilon_4 k, \tag{0.6}$$

enabling us to regard ± 1, $\pm i$, $\pm j$, $\pm k$ as elements of $(\mathbb{S}^3)'$ and, indeed, to think of x as a quaternion of unit norm although we do not divide through by the norm when considering expressions such as (0.6). If $x \in (\mathbb{S}^3)'$, this device also enables us to refer to $-x$ as the *antipode* of x. It is easy to check that $(\mathbb{S}^3)'$ has 80 distinct points. Since the function v referred to in 0.5 satisfies the equations

$$v(-x, y) = v(x, -y) = -v(x, y) \tag{0.7}$$

it can be described by means of a 40×40 multiplication table (see Section 2).

The table was compiled (and the function discovered) through the use of two QBASIC programs, one of which computes possible values of the function v satisfying the continuity (order-preservation) conditions and permitting an inductive computation of the rows and columns. The other program is a continuity checker that can be applied to a specified function v. Both of these were used in the process of arriving at the table listed. This turned out to be somewhat time consuming since the assignment may not be unique, in certain phases of the work multiple possibilities appear and there seems to be no way of predicting what choices will turn out to be part of a complete assignment. The continuity checker is reproduced in an Appendix. Further comments on the table are made in Section 2. In Section 1, as a preliminary, we discuss the one dimensional case treated by analogous methods. This leads to a different function to that referred to in (0.2) and described in [8], since it is of the form

$$m : \mathrm{op}(\mathbb{S}^1)' \times (\mathbb{S}^1)' \to \mathbb{S}^1. \tag{0.8}$$

This case is sufficiently simple that the continuity check (and compilation) can be done by inspection and a unicity property formulated.

The existence of multiplications of the type considered suggests that the spaces or posets concerned are examples of some notion of H-poset. We give a tentative definition in Section 3 and show that if a poset X has the relevant property then $|\mathscr{K}X|$ is an H-space. (The converse is true but the proof is not constructive.)

In Section 4 we return to the finite version of the *Hopf construction* introduced in [8], relating it to the classical case. In Section 5 we consider three examples of the construction and develop a notation for the points of the domain so that the model maps can conveniently be described.

1. The case of $\mathbb{S}^1$

The table for the 'multiplication' 0.4 given in [8] is as follows.

$$\begin{array}{c|cccccccc}
m & 1 & \cdot & i & \cdot & -1 & \cdot & -i & \cdot \\
\hline
1 & 1 & i & -1 & -1 & -1 & -i & 1 & 1 \\
\cdot & i & i & -1 & -i & -i & -i & 1 & i \\
i & -1 & -1 & -1 & -i & 1 & 1 & 1 & i \\
\cdot & -1 & -i & -i & -i & 1 & i & i & i \\
-1 & -1 & -i & 1 & 1 & 1 & i & -1 & -1 \\
\cdot & -i & -i & 1 & i & i & i & -1 & -i \\
-i & 1 & 1 & 1 & i & -1 & -1 & -1 & -i \\
\cdot & 1 & i & i & i & -1 & -i & -i & -i
\end{array} \tag{1.1}$$

To check the continuity (order-preservation) of m, note that the dots shown along the axes of the table refer to intermediate points of $(\mathbb{S}^1)'$ and that these are local suprema. Then it is sufficient to observe that in each row (respectively column) entries corresponding to intermediate points are greater than or equal to those corresponding to their immediate neighbours (regarded as points of $\mathbb{S}^1$). In [8], no attempt to account for the observation or to explain the existence of m was made. The emphasis there lay simply in using m to construct a finite model of Hopf's non-trivial homotopy class $S^3 \to S^2$. However, it will be convenient for us to consider these matters before embarking on the more complicated three-dimensional case.

The multiplication m exhibits a number of attractive algebraic properties. For example, it may be observed that m is commutative. It does not make sense to ask if m is associative unless we regard $\mathbb{S}^1$ as a subset of $(\mathbb{S}^1)'$ (it is not a subposet). If we do so then it is easy to check that the property fails. However, m *does* satisfy the analog of the property 0.7. Assuming this equation holds for m, we may describe the multiplication more economically in the following 4×4 table.

$$\begin{array}{r|cccc} m & 1 & i & 1+i & 1-i \\ \hline 1 & 1 & -1 & i & 1 \\ i & -1 & -1 & -1 & i \\ 1+i & i & -1 & i & i \\ 1-i & 1 & i & i & -i \end{array} \tag{1.2}$$

Note that the continuity of m may be checked also via table 1.2. Firstly the requirement of continuity places no restriction on the entries in the top left 2×2 square, since the points 1 and i are not related in $(\mathbb{S}^1)'$, but we do have $1 \leqslant 1+i$ and $i \leqslant 1+i$. It follows that in each row (and column) the entries corresponding to these must be similarly related in $\mathbb{S}^1$. A similar condition applies to the points 1 and $1-i$, but when we consider i and $1-i$, the condition is that (in each row and column) *minus* the entry corresponding to i has to be less than or equal to the entry corresponding to $1-i$.

In fact, we may regard the function m as a kind of continuous extension of the restriction of the multiplication to the 2×2 square. In this case the restriction is not particularly interesting. However, the situation is somewhat different if we consider the related multiplication of form

$$w : \mathrm{op}(\mathbb{S}^1)' \times (\mathbb{S}^1)' \to \mathbb{S}^1.$$

defined by the table

$$\begin{array}{r|cccc} w & 1 & i & 1+i & 1-i \\ \hline 1 & 1 & i & i & -i \\ i & i & -1 & i & i \\ 1+i & 1 & -1 & i & 1 \\ 1-i & 1 & 1 & 1 & -i \end{array}, \tag{1.3}$$

where it is understood that w satisfies Eq. (0.7) (with v replaced by w). The continuity of w can also be checked from table 1.3. As before, there is no restriction on the entries of the top left 2×2 square, but since the left factor poset now has the opposite order we have

$$1+i \leqslant 1, \;\; 1+i \leqslant i, \;\; 1-i \leqslant 1, \;\; 1-i \leqslant -i$$

hence in each column the entry corresponding to $1+i$ has to be *less than or equal to* the entry corresponding to 1, etc. The condition for rows, however, is unchanged. The reader will observe that the entries in table 1.3 (outside the top left 2×2 square) uniquely satisfy these conditions. It follows that table 1.3, yields the unique continuous extension of the restricted multiplication. Note that the restricted multiplication agrees exactly (on its domain) with the usual multiplication of complex numbers.

Using Eq. (0.7) we recover the full table for w:

$$\begin{array}{c|cccccccc}
w & 1 & \cdot & i & \cdot & -1 & \cdot & -i & \cdot \\
\hline
1 & 1 & i & i & i & -1 & -i & -i & -i \\
\cdot & 1 & i & -1 & -1 & -1 & -i & 1 & 1 \\
i & i & i & -1 & -i & -i & -i & 1 & i \\
\cdot & -1 & -1 & -1 & -i & 1 & 1 & 1 & i \\
-1 & -1 & -i & -i & -i & 1 & i & i & i \\
\cdot & -1 & -i & 1 & 1 & 1 & i & -1 & -1 \\
-i & -i & -i & 1 & i & i & i & -1 & -i \\
\cdot & 1 & 1 & 1 & i & -1 & -1 & -1 & -i
\end{array}. \tag{1.4}$$

Note that the full table exhibits the same zig-zag pattern as table 1.1 but it has been displaced (one row has been moved from the bottom of table 1.1 to the top).

The element 1 certainly does not behave as a strict identity element for the multiplication, nevertheless we can regard it as a left and right *homotopy identity*: observe that the top row of the table defines a map $(\mathbb{S}^1)' \to \mathbb{S}^1$ of degree 1 (and hence a weak homotopy equivalence), since it winds the 8-point circle just once around $\mathbb{S}^1$. A similar remark applies to the left column.

Note that the multiplication w is not commutative, however, in a sense that we will not make precise here, it is clearly *homotopy commutative*.

2. The case of $\mathbb{S}^3$

We have remarked in the introduction there are 80 points in the poset $(\mathbb{S}^3)'$. As in the previous case, to achieve economy in the description of the multiplication v, we give special attention (somewhat arbitrarily) to the 40 point subset $L \subseteq (\mathbb{S}^3)'$ consisting of chains of the form

$$\varepsilon_1 1 + \varepsilon_2 i + \varepsilon_3 j + \varepsilon_4 k,$$

in which the first non-zero coefficient is $+1$. Then $(\mathbb{S}^3)' = L \cup -L$, and L itself has the decomposition

$$L = L_1 \cup L_2 \cup L_3 \cup L_4,$$

where L_i $(1 \leqslant i \leqslant 4)$ comprises the points having exactly i non-zero coefficients. Thus we have

$$L_1 = \{1, i, j, k\}$$

$$L_2 = \{1+i,\ 1+j,\ 1+k,\ 1-i,\ 1-j,\ 1-k,\ i+j,\ i+k,\ j+k,$$
$$i-j,\ i-k,\ j-k\}$$

$$L_3 = L_{31} \cup L_{32},$$

where

$$L_{31} = \{i+j+k, i-j+k, i+j-k, i-j-k\}$$

and

$$L_{32} = \{1+i+j,\ 1+i+k,\ 1+j+k,\ 1+i-j,\ 1+i-k,\ 1+j-k,\\ 1-i+j,\ 1-i+k,\ 1-j+k,\ 1-i-j,\ 1-i-k,\ 1-j-k\},$$

$$L_4 = \{1+i+j+k,\ 1-i+j+k,\ 1+i-j+k,\ 1+i+j-k,\ 1-i-j+k,\\ 1-i+j-k,\ 1+i-j-k,\ 1-i-j-k\}.$$

We use the above decomposition to enumerate L, with its subsets as ordered above, and denote by x_n the nth point $(1 \leqslant n \leqslant 40)$. In this way we obtain

$x_1 = 1,$	$x_6 = 1+j,$	$x_{11} = i+j,$	$x_{16} = j-k,$	$x_{21} = 1+i+j,$	$x_{26} = 1+j-k,$	$x_{31} = 1-i-k,$	$x_{36} = 1+i+j-k$
$x_2 = i,$	$x_7 = 1+k,$	$x_{12} = i+k,$	$x_{17} = i+j+k,$	$x_{22} = 1+i+k,$	$x_{27} = 1-i+j,$	$x_{32} = 1-j-k,$	$x_{37} = 1-i-j+k$
$x_3 = j,$	$x_8 = 1-i,$	$x_{13} = j+k,$	$x_{18} = i-j+k,$	$x_{23} = 1+j+k,$	$x_{28} = 1-i+k,$	$x_{33} = 1+i+j+k,$	$x_{38} = 1-i+j-k$
$x_4 = k,$	$x_9 = 1-j,$	$x_{14} = i-j,$	$x_{19} = i+j-k,$	$x_{24} = 1+i-j,$	$x_{29} = 1-j+k,$	$x_{34} = 1-i+j+k,$	$x_{39} = 1+i-j-k$
$x_5 = 1+i,$	$x_{10} = 1-k,$	$x_{15} = i-k,$	$x_{20} = i-j-k,$	$x_{25} = 1+i-k,$	$x_{30} = 1-i-j,$	$x_{35} = 1+i-j+k,$	$x_{40} = 1-i-j-k.$

Since the poset of chains is ordered by inclusion, note that the points in each L_i are unrelated, but each point x of L_2 is above exactly two points of $L_1 \cup -L_1$, these are denoted D_1x and D_2x; each point x of L_3 is above exactly three points of $L_2 \cup -L_2$, these are denoted D_1x, D_2x and D_3x; each point x of L_4 is above exactly four points of $L_3 \cup -L_3$, these are denoted D_1x, D_2x, D_3x and D_4x.

Following the usage in 0.4 we call $\mathrm{op}(\mathbb{S}^3)' \times (1)$ and $(1) \times (\mathbb{S}^3)'$ the *axes* and describe the restrictions of v to the axes as the associated *axial maps* into $\mathbb{S}^3$.

Our main result is as follows.

Theorem 2.1. *There exists a continuous map* $v : \mathrm{op}(\mathbb{S}^3)' \times (\mathbb{S}^3)' \to \mathbb{S}^3$, *satisfying equations* (0.7), *whose associated axial maps* $f_1 : \mathrm{op}(\mathbb{S}^3)' \to \mathbb{S}^3$ *and* $f_2 : (\mathbb{S}^3)' \to \mathbb{S}^3$ *are weak homotopy equivalences, and which restricted to* $L_1 \times L_1$ *agrees with the table*

$$\begin{array}{c|cccc} v & 1 & i & j & k \\ \hline 1 & 1 & i & j & k \\ i & i & -1 & k & -j \\ j & j & -k & -1 & i \\ k & k & j & -i & -1 \end{array}.$$

Moreover $|\mathscr{K} f_i| \simeq 1 : S^3 \to S^3 \quad (i = 1,\ 2)$.

Table 1

v	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
1	1	i	j	k	i	j	k	$-i$	$-j$	$-k$	j	k	k	$-j$	$-k$	$-k$	k	k	$-k$	$-k$
2	i	-1	k	$-j$	i	k	$-j$	i	$-k$	j	k	$-j$	k	$-k$	j	k	k	$-k$	k	$-k$
3	j	$-k$	-1	i	$-k$	j	j	k	j	j	$-k$	$-k$	i	$-k$	$-k$	$-i$	$-k$	$-k$	$-k$	$-k$
4	k	j	$-i$	-1	k	k	k	k	k	k	j	j	$-i$	j	j	$-i$	j	j	j	j
5	1	-1	j	$-j$	i	j	$-j$	1	$-j$	j	j	$-j$	k	$-j$	j	j	k	$-j$	j	$-k$
6	1	i	-1	i	i	j	j	$-i$	1	j	i	i	i	i	$-k$	$-i$	i	i	$-k$	$-k$
7	1	i	$-i$	-1	i	j	k	$-i$	i	1	j	i	$-i$	i	j	$-i$	j	i	j	j
8	1	1	j	j	1	j	j	$-i$	$-j$	$-j$	j	j	j	$-j$	$-j$	$-k$	j	k	$-k$	$-j$
9	1	i	1	$-i$	i	1	$-j$	$-i$	$-j$	i	j	k	$-i$	i	i	i	k	k	j	i
10	1	i	i	1	i	j	1	$-i$	$-j$	$-k$	i	i	i	$-j$	i	i	i	$-j$	i	$-j$
11	i	-1	-1	i	i	i	i	i	j	j	-1	$-j$	i	$-k$	$-i$	$-i$	$-j$	$-k$	$-i$	$-k$
12	i	-1	$-i$	-1	i	k	i	i	i	j	$-i$	-1	$-i$	j	j	$-i$	$-i$	j	j	j
13	j	j	-1	-1	j	j	j	k	j	j	j	j	-1	j	j	$-i$	j	j	j	j
14	i	-1	1	$-i$	i	i	$-j$	i	i	i	k	$-j$	$-i$	-1	j	i	k	$-j$	k	j
15	i	-1	i	1	i	i	i	i	$-k$	i	$-j$	$-j$	i	$-j$	-1	i	$-j$	$-j$	$-j$	$-j$
16	j	$-j$	-1	1	$-k$	j	j	j	j	j	$-j$	$-j$	i	$-j$	$-j$	-1	$-j$	$-j$	$-j$	$-j$
17	i	-1	-1	-1	i	i	i	i	i	j	-1	-1	-1	j	$-i$	$-i$	-1	j	$-i$	j
18	i	-1	1	-1	i	i	i	i	i	i	$-i$	-1	$-i$	-1	j	1	$-i$	-1	j	j
19	i	-1	-1	1	i	i	i	i	j	i	-1	$-j$	i	$-j$	-1	-1	$-j$	$-j$	-1	$-j$
20	i	-1	1	1	i	i	i	i	i	i	$-j$	$-j$	1	-1	-1	i	$-j$	$-j$	$-j$	-1

In order to prove Theorem 2.1 it is, of course, sufficient to exhibit the complete table of a function v that satisfies the conditions. Note that a function v is continuous if, for each $(x,y)\in L\times L$ we have

$$v(x, D_i y) \leqslant v(x,y) \quad \text{and} \quad v(x,y) \leqslant v(D_j x, y) \tag{2.1}$$

whenever $D_i y$ and/or $D_j x$ are defined. There are a finite number of these conditions, and each is of a simple nature requiring only an inspection. Since many such inspections are needed, certain BASIC programs will be provided to enable the checking to be done quickly but, in principle, these are not part of the proof.

The axial map $f:(\mathbb{S}^3)'\to\mathbb{S}^3$ given by $f(y)=v(1,y)$, is necessarily order-preserving but it satisfies a stronger condition. Note by inspection, for each chain y, $f(y)=q$ only if q has non-zero coefficient in y. Bearing in mind Alexandroff's functor $\mathscr{K}$, it follows that q is one of the vertices of the simplex of $\mathscr{K}\mathbb{S}^3$ of which y is the barycentre. Hence $\mathscr{K}f$ is a *standard map* in the sense of [9, p. 35], hence $|\mathscr{K}f|$ is a homotopy equivalence by [9, Corollary 1.7.6], and f is a weak homotopy equivalence. Moreover, since $\mathscr{K}f$ is a simplicial approximation to the identity map $|\mathscr{K}X|\to|\mathscr{K}X|$, $|\mathscr{K}f|$ is homotopic to the identity map. A similar argument applies to the other axial map. This completes the argument for Theorem 2.1.

The full table, even for $L\times L$, is 40×40 and too large to be printed on a single page, we give instead the four 20×20 subtables (Tables 1–4).

Table 2

v	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40
1	j	k	k	$-j$	$-k$	$-k$	j	k	k	$-j$	$-k$	$-k$	k	k	k	$-k$	k	$-k$	$-k$	$-k$
2	k	$-j$	k	$-k$	j	k	k	$-j$	$-k$	$-k$	j	$-k$	k	k	$-k$	k	$-k$	k	$-k$	$-k$
3	$-k$	$-k$	j	$-k$	$-k$	j	k	k	j	k	k	j	$-k$	k	$-k$	$-k$	k	k	$-k$	k
4	k	k	k	k	k	k	k	k	k	k	k	k	k	k	k	k	k	k	k	k
5	j	$-j$	k	$-j$	j	j	j	$-j$	$-j$	$-j$	j	$-k$	k	k	$-j$	j	$-j$	j	$-k$	$-k$
6	j	j	j	i	$-k$	j	j	k	j	$-i$	j	j	j	k	j	$-k$	k	j	$-k$	j
7	j	k	k	i	j	j	j	k	k	$-j$	$-i$	i	k	k	k	j	k	j	j	$-j$
8	j	j	j	$-j$	$-j$	$-k$	j	j	k	$-j$	$-j$	$-j$	j	j	k	$-k$	k	$-k$	$-j$	$-j$
9	j	k	$-j$	$-j$	i	i	$-i$	$-j$	$-j$	$-j$	$-k$	$-j$	k	$-j$	k	j	$-j$	$-k$	$-j$	$-k$
10	j	i	j	$-j$	$-k$	$-k$	j	$-i$	$-j$	$-j$	$-k$	$-k$	j	j	$-j$	$-k$	$-j$	$-k$	$-k$	$-k$
11	i	$-j$	i	$-k$	j	j	k	i	j	j	j	j	$-j$	k	$-k$	j	j	k	$-k$	j
12	k	i	k	j	j	k	k	$-j$	i	i	j	j	k	k	j	k	$-j$	k	j	j
13	j	j	j	j	j	j	k	k	j	k	k	j	j	k	j	j	k	k	j	k
14	k	$-j$	$-j$	i	j	i	i	$-j$	$-j$	$-k$	j	i	k	$-j$	$-j$	k	$-k$	j	j	$-k$
15	$-j$	$-j$	i	$-k$	i	i	j	i	$-k$	$-k$	j	$-k$	$-j$	j	$-k$	$-j$	$-k$	j	$-k$	$-k$
16	$-k$	$-k$	j	$-k$	$-k$	j	j	j	j	j	j	j	$-k$	j	$-k$	$-k$	j	j	$-k$	j
17	i	i	i	j	j	j	k	i	i	i	j	j	i	k	j	j	i	k	j	j
18	k	i	$-j$	i	j	i	i	$-j$	i	i	i	i	k	$-j$	i	k	$-j$	i	j	i
19	i	$-j$	i	$-k$	i	i	j	i	j	j	j	j	$-j$	j	$-k$	i	j	j	$-k$	j
20	$-j$	$-j$	i	i	i	i	i	i	$-j$	$-k$	j	i	$-j$	i	$-j$	$-j$	$-k$	j	i	$-k$

Table 3

v	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
21	1	-1	-1	i	i	i	i	1	1	j	-1	i	i	i	$-i$	$-i$	i	i	$-i$	$-k$
22	1	-1	$-i$	-1	i	j	i	1	i	1	$-i$	-1	$-i$	i	j	$-i$	$-i$	i	j	j
23	1	i	-1	-1	i	j	j	$-i$	1	1	i	i	-1	i	j	$-i$	i	i	j	j
24	1	-1	1	$-i$	i	1	$-j$	1	i	i	j	$-j$	$-i$	-1	i	i	k	$-j$	j	i
25	1	-1	i	1	i	i	1	1	$-j$	i	i	i	i	$-j$	-1	i	i	$-j$	i	$-j$
26	1	i	-1	1	i	j	1	$-i$	1	j	i	i	i	i	i	-1	i	i	i	$-j$
27	1	1	-1	i	1	$-i$	j	$-i$	1	$-i$	i	i	i	1	$-j$	$-i$	i	i	$-k$	$-j$
28	1	1	$-i$	-1	1	$-i$	$-i$	$-i$	i	1	j	i	$-i$	i	1	$-i$	j	i	j	i
29	1	i	1	-1	i	1	$-j$	$-i$	i	1	j	i	$-i$	i	i	1	j	i	j	i
30	1	1	1	$-i$	1	1	$-i$	$-i$	$-j$	i	1	j	$-i$	i	i	i	j	k	i	i
31	1	1	i	1	1	j	1	$-i$	$-i$	$-j$	i	1	i	$-j$	i	i	i	$-j$	i	$-j$
32	1	i	1	1	i	1	1	$-i$	$-j$	i	i	i	1	i	i	i	i	$-j$	i	i
33	1	-1	-1	-1	i	i	i	1	1	1	-1	-1	-1	i	$-i$	$-i$	-1	i	$-i$	j
34	1	1	-1	-1	1	$-i$	$-i$	$-i$	1	1	i	i	-1	1	1	$-i$	i	i	j	1
35	1	-1	1	-1	i	1	i	1	i	1	$-i$	-1	$-i$	-1	i	1	$-i$	-1	j	i
36	1	-1	-1	1	i	i	1	1	1	i	-1	i	i	i	-1	-1	i	i	-1	$-j$
37	1	1	1	-1	1	1	$-i$	$-i$	i	1	1	i	$-i$	i	1	1	j	i	1	i
38	1	1	-1	1	1	$-i$	1	$-i$	1	$-i$	i	1	i	1	i	-1	i	1	i	$-j$
39	1	-1	1	1	i	1	1	1	i	i	i	i	1	-1	-1	i	i	$-j$	i	-1
40	1	1	1	1	1	1	1	$-i$	$-i$	i	1	1	1	i	i	i	1	$-j$	i	i

Table 4

v	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40
21	i	i	i	i	j	j	j	i	i	1	j	j	i	k	i	j	i	j	$-k$	j
22	j	i	k	i	j	j	j	$-j$	i	i	1	i	k	k	i	j	$-j$	j	j	i
23	j	j	j	i	j	j	j	k	j	$-i$	$-i$	1	j	k	j	j	k	j	j	$-i$
24	j	$-j$	$-j$	i	i	i	1	$-j$	$-j$	$-j$	j	i	k	$-j$	$-j$	j	$-j$	j	i	$-k$
25	i	i	i	$-j$	i	i	j	1	$-j$	$-j$	j	$-k$	i	j	$-j$	i	$-j$	j	$-k$	$-k$
26	j	i	j	i	$-k$	j	j	$-i$	1	$-i$	j	j	j	j	i	$-k$	$-i$	j	$-k$	j
27	j	j	j	1	$-j$	$-i$	$-i$	j	j	$-i$	$-i$	$-i$	j	j	j	$-k$	k	$-i$	$-j$	$-i$
28	j	j	$-i$	i	1	$-i$	$-i$	$-i$	k	$-j$	$-i$	i	j	$-i$	k	j	k	$-i$	i	$-j$
29	j	k	$-j$	i	i	1	$-i$	$-j$	$-j$	$-j$	$-i$	i	k	$-j$	k	j	$-j$	$-i$	i	$-j$
30	1	j	$-i$	$-j$	i	i	$-i$	$-i$	$-j$	$-j$	$-j$	$-j$	j	$-i$	k	i	$-j$	$-k$	$-j$	$-j$
31	j	1	j	$-j$	$-j$	$-k$	j	$-i$	$-i$	$-i$	$-j$	$-j$	j	j	$-j$	$-k$	$-i$	$-k$	$-j$	$-j$
32	i	i	1	$-j$	i	i	$-i$	$-i$	$-j$	$-j$	$-k$	$-j$	i	$-i$	$-j$	i	$-j$	$-k$	$-j$	$-k$
33	i	i	i	i	j	j	j	i	i	1	1	1	i	k	i	j	i	j	j	1
34	j	j	$-i$	1	1	$-i$	$-i$	$-i$	j	$-i$	$-i$	1	j	$-i$	j	j	k	$-i$	1	$-i$
35	j	i	$-j$	i	i	1	1	$-j$	i	i	1	i	k	$-j$	i	j	$-j$	1	i	i
36	i	i	i	i	i	i	j	1	1	1	j	j	i	j	i	i	1	j	$-k$	j
37	1	j	$-i$	i	1	1	$-i$	$-i$	$-j$	$-j$	$-i$	i	j	$-i$	k	1	$-j$	$-i$	i	$-j$
38	j	1	j	1	$-j$	$-i$	$-i$	$-i$	1	$-i$	$-i$	$-i$	j	j	1	$-k$	$-i$	$-i$	$-j$	$-i$
39	i	i	1	i	i	i	1	1	$-j$	$-j$	j	i	i	1	$-j$	i	$-j$	j	i	$-k$
40	1	1	1	$-j$	i	i	$-i$	$-i$	$-i$	$-i$	$-j$	$-j$	1	$-i$	$-j$	i	$-i$	$-k$	$-j$	$-j$

Remark 2.2. At first glance the Tables 1–4 do not appear to exhibit much regularity. This is perhaps an effect due to the somewhat arbitrary linear ordering of the points of L. However, the full 80×80 table exhibits remarkable statistical regularity. For example, if the numbers of entries are counted we find the following.

1	−1	i	$-i$	j	$-j$	k	$-k$
544	544	1056	1056	1056	1056	544	544

The table is essentially a compendium of a large number of 4×4 subsquares of the following type. Consider, for example, two typical chains of $\mathrm{op}(\mathbb{S}^3)'$ and of $(\mathbb{S}^3)'$, respectively:

$$c_1 = (1 - i + j - k, 1 + j - k, 1 - k, -k) \quad \text{and}$$
$$c_2 = (i, i + j, i + j + k, 1 + i + j + k).$$

Note that the elements of $(\mathbb{S}^3)'$ listed in c_1 are numbered 38, 26, 10 and −4. (Note the use of the minus sign for an element of $-L$.) Similarly the elements listed in c_2 are numbered 2, 11, 17 and 33. From the main table we can extract the entries for the 4×4 subtable

$$\begin{array}{r|cccc} v & 2 & 11 & 17 & 33 \\ \hline 4 & j & j & j & k \\ 10 & i & i & i & j \\ 26 & i & i & i & j \\ 38 & 1 & i & i & j \end{array} \tag{2.2}$$

The continuity conditions for v in respect of this selection of elements can be checked by inspection of the 4×4 subtable. In principle, the whole check could be conducted in this way, extracting for each pair of chains the appropriate 4×4 table. One might be tempted to think that the construction of the function v could be achieved by such piecemeal assignments, but a moment's thought shows this not to be the case: the relevant subtables overlap and have to be consistent with one another. Note that table (2.2) specifies the values of the multiplication function v when its domain is restricted to the product of two principal simplexes. The smooth appearance of such tables shows that v exhibits at least a local regularity.

Somewhat more regularity is in evidence in the $(33,40)\times(33,40)$ subsquare. Since this square describes the restriction of the multiplication to the barycentres of the principal simplexes of $\mathbb{S}^3$, we may regard it as a kind of Poincaré dual of the $(1,4)\times(1,4)$ square. (Compare the case $n=1$ in 1.3.) The 1-entries in the diagonal of positive slope (of the $(33,40)\times(33,40)$ square) are rather striking. These correspond to v products of conjugate quaternions and one can check the following by inspection of the tables.

Corollary 2.3. *If $x=\varepsilon_1 1+\varepsilon_2 i+\varepsilon_3 j+\varepsilon_4 k$, let $\bar{x}=\varepsilon_1 1-\varepsilon_2 i-\varepsilon_3 j-\varepsilon_4 k$. Then the function v satisfies the identity*

$$v(x,\bar{x}) = 1 \quad (x \in (\mathbb{S}^3)').$$

We recall [12] that a map $\mu: X\times Y \to Z$ is a *pairing* with *axes* $f: X\to Z$ and $g: Y\to Z$ if it satisfies

$$\mu \,|\, X \vee Y \simeq \triangledown_Z \circ (f \vee g): X \vee Y \to Z,$$

where $\simeq$ denotes the homotopy relation and $\triangledown_Z$ the codiagonal map $Z\vee Z\to Z$. Then we may record:

Corollary 2.4. *$v:\mathrm{op}(\mathbb{S}^3)'\times(\mathbb{S}^3)'\to\mathbb{S}^3$ is a pairing with axes $\mathrm{op}(\mathbb{S}^3)'\to\mathbb{S}^3$ and $(\mathbb{S}^3)'\to\mathbb{S}^3$.*

The existence of such a pairing was conjectured in [8].

3. *H*-space and *H*-poset

Let 1 be a preferred element of the finite poset X and let $X^{(n)}$ denote the nth barycentric subdivision of X, where $n\geqslant 0$. It is well-known that $|\mathscr{K}X|$, $|\mathscr{K}(\mathrm{op}X)|$ and $|\mathscr{K}X'|$ have homeomorphic underlying polyhedra. It will be convenient to identify $|\mathscr{K}X|$, $|\mathscr{K}X^{(n)}|$ and $|\mathscr{K}(\mathrm{op}X^{(n)})|$ via this homeomorphism. An order-preserving function

$$w:(X\times X)^{(n)} \to X$$

is a *weak pairing with axes*

$$X^{(n)} \xrightarrow{i_1^{(n)}} (X\times X)^{(n)} \xrightarrow{w} X, \qquad X^{(n)} \xrightarrow{i_2^{(n)}} (X\times X)^{(n)} \xrightarrow{w} X,$$

where $i_1: X\to X\times X$, $i_2: X\to X\times X$ are such that $i_1x=(x,1)$ and $i_2x=(1,x)$ $(x\in X)$.

We say that X is an *H-poset* if, for some $n\geqslant 0$, X admits a pairing $\mathrm{op}X^{(n)}\times X^{(n)}\to X$ or a pairing $X^{(n)}\times X^{(n)}\to X$ or a weak pairing with axes f_1, f_2 such that $|\mathscr{K}f_1|$ and $|\mathscr{K}f_2|$ are homotopic to the identity map $|\mathscr{K}X|\to|\mathscr{K}X|$.

We recall that a space W, with preferred element 1, is an *H-space* if there exists a map $m: W\times W\to W$ and a homotopy-commutative diagram

$$\begin{tikzcd} W \arrow[d, "i_1"'] \arrow[dr, "1_W"] & \\ W\times W \arrow[r, "m"] \arrow[d, "i_2"'] & W \\ W \arrow[ur, "1_W"'] & \end{tikzcd} ,$$

where $i_1 w=(w,1)$, $i_2 w=(1,w)$ $(w\in W)$.

Proposition 3.1. *If X is an H-poset then $|\mathscr{K}X|$ is an H-space.*

The proof of the Proposition requires a lemma.

Lemma 3.2. *There exists a diagram*

$$\begin{tikzcd} {|\mathcal{K}(X\times Y)|} \arrow[rr, "h"] \arrow[dr, "q(X\times Y)"'] & & {|\mathcal{K}X|\times|\mathcal{K}Y|} \arrow[dl, "qX\times qY"] \\ & X\times Y & \end{tikzcd}$$

in which h is a homeomorphism, natural in X and Y.

Proof. The map h is induced by the diagram of projections

$$\begin{tikzcd} {|\mathcal{K}X|} \arrow[d, "qX"'] & {|\mathcal{K}(X\times Y)|} \arrow[l, "|\mathcal{K}\pi_1|"'] \arrow[r, "|\mathcal{K}\pi_2|"] \arrow[d, "q(X\times Y)"] & {|\mathcal{K}Y|} \arrow[d, "qY"] \\ X & X\times Y \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] & Y \end{tikzcd}$$

and hence is natural in X and Y. If we consider a particular point of $|\mathscr{K}X|\times|\mathscr{K}Y|$, we may note that it is a pair (x,y), where $x\in|\sigma|$, $y\in|\tau|$ and σ, τ are simplexes of $\mathscr{K}X$, $\mathscr{K}Y$ respectively. In view of the naturality, it will be sufficient to consider the case in which X and Y are totally ordered sets. Then $|\mathscr{K}X|$ and $|\mathscr{K}Y|$ may be considered to be convex subspaces of appropriate Euclidean spaces E and E', respectively, and indeed as the convex hulls of the sets V_X, V_Y of vertices of $|\mathscr{K}X|$ and $|\mathscr{K}Y|$. respectively. Then we claim that $|\mathscr{K}(X\times Y)|$ is homeomorphic with the convex hull of $V_X\times V_Y$ in $E\times E'$ and in turn with $|\mathscr{K}X|\times|\mathscr{K}Y|$, completing the proof.

Proof of Proposition 3.1. Suppose that X admits a pairing $v:\mathrm{op}X^{(n)}\times X^{(n)}\to X$ with axes $f_1:\mathrm{op}X^{(n)}\to X, f_2: X^{(n)}\to X$, where $|\mathscr{K}f_1|\simeq 1:|\mathscr{K}(\mathrm{op}X^{(n)})|=|\mathscr{K}X|\to|\mathscr{K}X|$

and similarly $|\mathscr{K}f_2|\simeq 1$. Then we have a commutative diagram of:

$$\begin{tikzcd}
{|\mathcal{K}(\mathrm{op}X^{(n)}\times(1))|} \arrow[r, "\approx"] \arrow[d, "|\mathcal{K}i_1|"'] & {|\mathcal{K}X|\times(1)} \arrow[d, "j_1"] \\
{|\mathcal{K}(\mathrm{op}X^{(n)}\times X^{(n)})|} \arrow[r, "h"] & {|\mathcal{K}X|\times|\mathcal{K}X|}
\end{tikzcd}.$$

Since $|\mathscr{K}v|\circ|\mathscr{K}i_1|=|\mathscr{K}(v\circ i_1)|\simeq 1$, and similarly for i_2, we have a homotopy commutative diagram

$$\begin{tikzcd}
{|\mathcal{K}X|} \arrow[d, "j_1"'] \arrow[dr, "\mathrm{id}"] & \\
{|\mathcal{K}X|\times|\mathcal{K}X|} \arrow[r, "|\mathcal{K}v|h^{-1}"] & {|\mathcal{K}X|} \\
{|\mathcal{K}X|} \arrow[u, "j_2"] \arrow[ur, "\mathrm{id}"'] &
\end{tikzcd},$$

so that $|\mathscr{K}X|$ is an H-space. The argument in the case of a pairing $X^{(n)}\times X^{(n)}\to X$ is similar. Suppose then that we have a weak pairing $(X\times X)^{(n)}\to X$. Then a diagram of type

$$\begin{tikzcd}
{|\mathcal{K}X|} \arrow[r] \arrow[d] & {|\mathcal{K}X\times(1)|} \arrow[r, "\approx"] \arrow[d] & {|\mathcal{K}(X\times(1))^{(n)}|} \arrow[d] \arrow[dr] & \\
{|\mathcal{K}X|\times|\mathcal{K}X|} \arrow[r, "h^{-1}"] & {|\mathcal{K}(X\times X)|} \arrow[r, "\approx"] & {|\mathcal{K}(X\times X)^{(n)}|} \arrow[r, "|\mathcal{K}w|"] & {|\mathcal{K}X|} \\
{|\mathcal{K}X|} \arrow[r] \arrow[u] & {|\mathcal{K}((1)\times X)|} \arrow[r, "\approx"] \arrow[u] & {|\mathcal{K}((1)\times X)^{(n)}|} \arrow[u] \arrow[ur] &
\end{tikzcd}$$

yields the desired conclusion. This completes the proof. □

Remark 3.3. The converse of Proposition 3.1 also holds but the proof depends on the classical simplicial approximation theorem and hence is non-constructive.

4. Homotopy pushout and Hopf construction

If X is a poset the *non-Hausdorff cone*, $\mathbb{C}X=(X,\hat{x})$, is the poset X equipped with one additional point $\hat{x}$ as upper bound. The *non-Hausdorff suspension*, $\mathbb{S}X=(X,\hat{n},\hat{s})$, is the union of two copies of $\mathbb{C}X$ whose intersection is X. These basic concepts go back to McCord's paper [10]. They are special cases of the following notion of non-Hausdorff double mapping cylinder. As far as we are aware this first appeared in the literature in [17] (in not quite identical form).

Let $f_1:A\to X$, $f_2:A\to Y$ be a pair of poset maps. Then $\mathbb{M}(f_1,f_2)$, the *non-Hausdorff double mapping cylinder* of f_1 and f_2, is the poset obtained from the disjoint union of (finite) posets $X+A+Y$ by specifying the additional relations $a\leqslant f_1(a)$, $a\leqslant f_2(a)$ for all $a\in A$.

The construction is functorial: a commutative diagram of poset maps

$$\begin{tikzcd} X_1 \arrow[d, "x"'] & A_1 \arrow[l, "f_1"'] \arrow[r, "f_2"] \arrow[d, "a"'] & Y_1 \arrow[d, "y"] \\ X_2 & A_2 \arrow[l, "g_1"] \arrow[r, "g_2"'] & Y_2 \end{tikzcd} \tag{4.1}$$

induces a map $\mathbb{M}(x,a,y):\mathbb{M}(f_1,f_2)\to\mathbb{M}(g_1,g_2)$.

We shall be particularly interested in the following special cases:

(4.2a) The *non-Hausdorff join* of the posets X and Y

$$X \circledast Y = \mathbb{M}(\pi_1,\pi_2),$$

where $\pi_1 : X\times Y\to X$ and $\pi_2 : X\times Y\to Y$ denote the two projections;

(4.2b) The non-Hausdorff suspension of Z

$$\mathbb{S}Z = \mathbb{M}(n,s),$$

where $n:Z\to\hat{n}$ and $s:Z\to\hat{s}$ are constant maps from Z to the singleton posets $\hat{n}$ and $\hat{s}$.

Given a pairing $v:X\times Y\to Z$, the *Hopf construction* of v is the map

$$\Gamma(v):X \circledast Y \to \mathbb{S}Z \tag{4.3}$$

induced by the diagram:

$$\begin{tikzcd} X \arrow[d, "n"'] & X\times Y \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] \arrow[d, "v"] & Y \arrow[d, "s"] \\ \hat{n} & Z \arrow[l] \arrow[r] & \hat{s} \end{tikzcd}.$$

We wish to relate $\Gamma(v)$ to the *generalised Hopf construction*, as defined by G.W. Whitehead [16, p. 502], of the associated pairing of polyhedra $|\mathscr{K}v|h^{-1}$, where h is the map defined in Lemma 3.2.

The key to understanding these constructions is to regard the non-Hausdorff double mapping cylinder as (part of) the poset analog of the standard homotopy pushout construction for spaces. Note that, given surjective, order-preserving functions $f_1:A\to X$, $f_2:A\to Y$, we have a diagram

$$\begin{tikzcd} A \arrow[r, "f_2"] \arrow[d, "f_1"'] & Y \arrow[d] \\ X \arrow[r] & \mathbb{M}(f_1,f_2) \end{tikzcd}, \tag{4.3a}$$

which we may regard as commuting via a ‘homotopy’, where the closed unit interval I is replaced by the three point poset

$$\mathbb{I} = 0 \leftarrow 1/2 \rightarrow 1.$$

Moreover the reader will observe that if we apply the functor $|\mathscr{K}-|$ to diagram 4.3a, we obtain

$$\begin{tikzcd} |\mathcal{K}A| \arrow[r, "|\mathcal{K}f_2|"] \arrow[d, "|\mathcal{K}f_1|"] & |\mathcal{K}Y| \arrow[d] \\ |\mathcal{K}X| \arrow[r] & M \end{tikzcd},$$

where the space M obtained is exactly the ordinary double mapping cylinder, in the category of spaces, of the maps $\mathscr{K}f_1$ and $\mathscr{K}f_2$ and, indeed, the diagram obtained is (up to natural homeomorphism) the standard homotopy pushout diagram associated with I. We have in fact verified that the functor $|\mathscr{K}-|$ preserves this type of homotopy pushout.

In particular we can deduce that

$$|\mathscr{K}(X \circledast Y)| \approx |\mathscr{K}X| * |\mathscr{K}Y|, \tag{4.3b}$$

where $P*Q$ refers to the classical *join* construction of polyhedra, since it is a consequence of the definition of the join that the diagram

$$\begin{tikzcd} P \times Q \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & Q \arrow[d] \\ P \arrow[r] & P * Q \end{tikzcd}$$

is an instance of the standard homotopy pushout.

Proposition 4.1. *The homotopy class of* $|\mathscr{K}\Gamma(v)|$ *is equal to the class of the generalised Hopf construction of* $|\mathscr{K}v|h^{-1}$.

Proof. It is sufficient to observe that the maps shown in the diagram

$$\begin{tikzcd} |\mathcal{K}X| \arrow[d, "1"'] & |\mathcal{K}X| \times |\mathcal{K}Y| \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] \arrow[d, "h^{-1}"] & |\mathcal{K}Y| \arrow[d, "1"] \\ |\mathcal{K}X| \arrow[d] & |\mathcal{K}(X \times Y)| \arrow[l, "\mathcal{K}\pi_1"'] \arrow[r, "\mathcal{K}\pi_2"] \arrow[d, "|\mathcal{K}v|"] & |\mathcal{K}Y| \arrow[d] \\ |\hat{n}| & |\mathcal{K}Z| \arrow[l, "\mathcal{K}n|"'] \arrow[r, "|\mathcal{K}s|"] & |\hat{s}| \end{tikzcd}$$

induce, via homotopy pushout, the maps

$$|\mathscr{K}X| * |\mathscr{K}Y| \to |\mathscr{K}(X \circledast Y)| \to |\mathscr{K}\mathbb{S}Z| \tag{4.4}$$

whose composite agrees with the desired generalised Hopf construction and of which the first is a homotopy equivalence. Note that $|\mathscr{K}\mathbb{S}Z|$ coincides with the unreduced suspension of $|\mathscr{K}Z|$. Descriptions of the join and the unreduced suspension are given in [16].

Remark 4.2. There is an alternative construction of double mapping cylinder associated with a pair of (surjective) poset maps $f_1 : A \to X$, $f_2 : A \to Y$. We denote by $\mathbb{M}^o(f_1, f_2)$ the poset obtained from the disjoint union $X + A + Y$ by specifying additional relations $a \geqslant f_1(a)$, $a \geqslant f_2(a)$. Then there is also a diagram corresponding to (4.3a) with $\mathbb{M}^o(f_1, f_2)$ replacing $\mathbb{M}(f_1, f_2)$. This diagram is also a 'standard homotopy pushout', being associated with the unit interval object $\mathbb{I}^{\mathrm{op}}$. The corresponding notion of non-Hausdorff join, namely

$$X \circledast^o Y = \mathbb{M}^o(\pi_1, \pi_2) \tag{4.5}$$

enjoys properties similar to the $\circledast$ join. In particular there is a homeomorphism analogous to (4.3b) as well as an alternative version of the Hopf construction. We shall see that these constructions are sometimes useful to capture models involving particular posets.

5. Examples and applications

In this section we examine iterated non-Hausdorff joins of the discrete two-point space S^0, identifying the posets obtained and developing a notation for their elements with a view to studying the Hopf construction maps associated with the known examples of pairings. We begin by establishing the following relation.

$$S^0 \circledast S^0 \approx \mathrm{op}(\mathbb{S}^1)'. \tag{5.1}$$

First note that in the definition of the non-Hausdorff double mapping cylinder (see diagram 4.3a) the spaces X and Y become subposets of the cylinder produced. For this reason we use the notation $\{1, -1\}$ and $\{i, -i\}$ for the elements of the left and right copies of S^0 in 5.1. Then the elements of the poset $S^0 + S^0{\times}S^0 + S^0$ and the order relations are as indicated in the diagram

$$\begin{tikzcd} & i & -i \\ 1 & (1,i) \arrow[l] \arrow[u] & (1,-i) \arrow[u] \arrow[ll, bend left] \\ -1 & (-1,i) \arrow[l] \arrow[uu, bend right] & (-1,-i) \arrow[ll, bend left] \arrow[uu, bend right] \end{tikzcd} \quad = \quad \begin{tikzcd} & i \arrow[d, equal] & -i \arrow[d, equal] \\ 1 \arrow[r, equal] & (1,i) & (1,-i) \\ -1 \arrow[r, equal] & (-1,i) & (-1,-i) \end{tikzcd},$$

where the version on the right is to be understood as an abbreviation. (Specifically the vertical double lines stand for the pairs of arrows into i and $-i$ from the points below them and the horizontal double lines stand for the pairs of horizontal arrows into 1 and -1 from the points to their right.) Altering the positions of the points and changing the notation for ordered pairs from bracket to sum the diagram yields

$$\begin{tikzcd}
-1+i \arrow[r] \arrow[d] & i & 1+i \arrow[l] \arrow[d] \\
-1 & & 1 \\
-1-i \arrow[r] \arrow[u] & -i & 1-i \arrow[l] \arrow[u]
\end{tikzcd}
\quad = \mathrm{op}(\mathbb{S}^1)'.$$

Note that in the process the elements of the respective copies of S^0 have become antipodal. In a similar way we can represent $(S^0 \circledast S^0) \circledast S^0$ as

$$\begin{tikzcd}[column sep=small]
& & & & j & & & & \\
1+j \arrow[urrrr] \arrow[d] & 1-i+j \arrow[l] \arrow[r] \arrow[urrr] \arrow[d] & -i+j \arrow[urr] \arrow[d] & -1-i+j \arrow[l] \arrow[r] \arrow[ur] \arrow[d] & -1+j \arrow[u] \arrow[d] & -1+i+j \arrow[l] \arrow[r] \arrow[ul] \arrow[d] & i+j \arrow[ull] \arrow[d] & 1+i+j \arrow[l] \arrow[r] \arrow[ulll] \arrow[d] & 1+j \arrow[ullll] \arrow[d] \\
1 & 1-i \arrow[l] \arrow[r] & -i & -1-i \arrow[l] \arrow[r] & -1 & -1+i \arrow[l] \arrow[r] & i & 1+i \arrow[l] \arrow[r] & 1 \\
1-j \arrow[drrrr] \arrow[u] & 1-i-j \arrow[l] \arrow[r] \arrow[drrr] \arrow[u] & -i-j \arrow[drr] \arrow[u] & -1-i-j \arrow[l] \arrow[r] \arrow[dr] \arrow[u] & -1-j \arrow[d] \arrow[u] & -1+i-j \arrow[l] \arrow[r] \arrow[dl] \arrow[u] & i-j \arrow[dll] \arrow[u] & 1+i-j \arrow[l] \arrow[r] \arrow[dlll] \arrow[u] & 1-j \arrow[dllll] \arrow[u] \\
& & & & -j & & & &
\end{tikzcd} .$$

Here it is understood that the points at the extremes of the second, third and fourth rows have to be identified. Alternatively, $S^0 \circledast (S^0 \circledast S^0)$ can be displayed:

$$\begin{tikzcd}[column sep=small]
& i \arrow[d, equal] & i+j \arrow[l] \arrow[r] \arrow[d, equal] & j \arrow[d, equal] & -i+j \arrow[l] \arrow[r] \arrow[d, equal] & -i \arrow[d, equal] & -i-j \arrow[l] \arrow[r] \arrow[d, equal] & -j \arrow[d, equal] & i-j \arrow[l] \arrow[r] \arrow[d, equal] & {} \\
1 \arrow[r, equal] & 1+i & 1+i+j \arrow[l] \arrow[r] & 1+j & 1-i+j \arrow[l] \arrow[r] & 1-i & 1-i-j \arrow[l] \arrow[r] & 1-j & 1+i-j \arrow[l] \arrow[r] & {} \\
-1 \arrow[r, equal] & -1+i & -1+i+j \arrow[l] \arrow[r] & -1+j & -1-i+j \arrow[l] \arrow[r] & -1-i & -1-i-j \arrow[l] \arrow[r] & -1-j & -1+i-j \arrow[l] \arrow[r] & {}
\end{tikzcd}$$

Note that the horizontal double lines to 1, (respectively, -1) stand for sheaves of arrows from *each* of the points to their right. The arrows pointing outward on the extreme right should be regarded as being directed to points in the same row seven places to the left, so as to complete eight-point circles. It is evident that

$$(S^0 \circledast S^0) \circledast S^0 = S^0 \circledast (S^0 \circledast S^0) = \mathrm{op}(\mathbb{S}^2)'. \tag{5.2}$$

More surprisingly, perhaps, we have

$$
\mathrm{op}(\mathbb{S}^1)' \circledast \mathrm{op}(\mathbb{S}^1)' \quad = \quad \begin{array}{ccccc} -1+i & \rightarrow & i & \leftarrow & 1+i \\ \downarrow & & & & \downarrow \\ -1 & & & & 1 \\ \uparrow & & & & \uparrow \\ -1-i & \rightarrow & -i & \leftarrow & 1-i \end{array} \quad \circledast \quad \begin{array}{ccccc} -j+k & \rightarrow & k & \leftarrow & j+k \\ \downarrow & & & & \downarrow \\ -j & & & & j \\ \uparrow & & & & \uparrow \\ -j-k & \rightarrow & -k & \leftarrow & j-k \end{array} \quad =
$$

$$
\begin{array}{cccccccccccccccccc}
 & & j & \longleftarrow & j+k & \longrightarrow & k & \longleftarrow & -j+k & \longrightarrow & -j & \longleftarrow & -j-k & \longrightarrow & -k & \longleftarrow & j-k & \longrightarrow \\
 & & \| & & \| & & \| & & \| & & \| & & \| & & \| & & \| & \\
1 & = & 1+j & \leftarrow & 1+j+k & \rightarrow & 1+k & \leftarrow & 1-j+k & \rightarrow & 1-j & \leftarrow & 1-j-k & \rightarrow & 1-k & \leftarrow & 1+j-k & \rightarrow \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & \\
1+i & = & 1+i+j & \leftarrow & 1+i+j+k & \rightarrow & 1+i+k & \leftarrow & 1+i-j+k & \rightarrow & 1+i-j & \leftarrow & 1+i-j-k & \rightarrow & 1+i-k & \leftarrow & 1+i+j-k & \rightarrow \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
i & = & i+j & \leftarrow & i+j+k & \rightarrow & i+k & \leftarrow & i-j+k & \rightarrow & i-j & \leftarrow & i-j-k & \rightarrow & i-k & \leftarrow & i+j-k & \rightarrow \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & \\
-1+i & = & -1+i+j & \leftarrow & -1+i+j+k & \rightarrow & -1+i+k & \leftarrow & -1+i-j+k & \rightarrow & -1+i-j & \leftarrow & -1+i-j-k & \rightarrow & -1+i-k & \leftarrow & -1+i+j-k & \rightarrow \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
-1 & = & -1+j & \leftarrow & -1+j+k & \rightarrow & -1+k & \leftarrow & -1-j+k & \rightarrow & -1-j & \leftarrow & -1-j-k & \rightarrow & -1-k & \leftarrow & -1+j-k & \rightarrow \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & \\
-1-i & = & -1-i+j & \leftarrow & -1-i+j+k & \rightarrow & -1-i+k & \leftarrow & -1-i-j+k & \rightarrow & -1-i-j & \leftarrow & -1-i-j-k & \rightarrow & -1-i-k & \leftarrow & -1-i+j-k & \rightarrow \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
-i & = & -i+j & \leftarrow & -i+j+k & \rightarrow & -i+k & \leftarrow & -i-j+k & \rightarrow & -i-j & \leftarrow & -i-j-k & \rightarrow & -i-k & \leftarrow & -i+j-k & \rightarrow \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & \\
1-i & = & 1-i+j & \leftarrow & 1-i+j+k & \rightarrow & 1-i+k & \leftarrow & 1-i-j+k & \rightarrow & 1-i-j & \leftarrow & 1-i-j-k & \rightarrow & 1-i-k & \leftarrow & 1-i+j-k & \rightarrow \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow &
\end{array}
\tag{5.3}
$$

which (interpreted in terms of the conventions explained above) we may recognise as the poset $\mathrm{op}(\mathbb{S}^3)'$. We have therefore established the following isomorphism:

$$\mathrm{op}(\mathbb{S}^1)' \circledast \mathrm{op}(\mathbb{S}^1)' \approx \mathrm{op}(\mathbb{S}^3)' \tag{5.4a}$$

Remark 4.2 enables us to express the isomorphism 5.4a in terms of posets with the opposite partial order, yielding

$$(\mathbb{S}^1)' \circledast^o (\mathbb{S}^1)' \approx (\mathbb{S}^3)'. \tag{5.4b}$$

We conclude with some examples of the Hopf construction at the poset level.

Our first example, included mainly as a simple illustration of the algorithm, is the case of the multiplication on the (discrete) two-point space S^0. If we take $S^0 = \{1, -1\}$ then the following multiplication defines a pairing with axes the identity map $S^0 \to S^0$:

m	1	-1
1	1	-1
-1	-1	1

(5.5)

To write down $\Gamma(m)$ we first restate the table replacing the notation for the points of the second factor:

m	i	$-i$
1	1	-1
-1	-1	1

This tells us that $\Gamma(m)$ restricted to the product of spheres yields the function (into $\mathbb{S}^1$):

$$1+i \mapsto 1, \;\; 1-i \mapsto -1, \;\; -1+i \mapsto -1, \;\; -1-i \mapsto 1. \tag{5.5a}$$

(Note that we have changed from ordered pair to sum notation.) Moreover, recalling from (4.3) that the points of the left hand factor space are sent into $\hat{n}=i$ and the points of the right hand factor space into $\hat{s}=-i$, we may complete the assignment thus:

$$1 \mapsto i, \;\; -1 \mapsto i, \;\; i \mapsto -i, \;\; -i \mapsto -i. \tag{5.5b}$$

It can now be checked that, as defined, $\Gamma(m)$ is an order-preserving function and that it corresponds (when we orient the circles via anticlockwise rotation) to a map of degree -2 from $\mathrm{op}(\mathbb{S}^1)'$ to $\mathbb{S}^1$.

It has, of course, been known for some time that a degree two class $S^1 \to S^1$ can be obtained by applying Hopf's construction to a multiplication on the 0-dimensional sphere cf. [3], however it remains of some interest to note that it can be done at the poset level.

In our next example we use the multiplication 0.4 specified in table 1.1 to construct a map

$$\Gamma : (\mathbb{S}^3)' \approx (\mathbb{S}^1)' \circledast^o (\mathbb{S}^1)' \to \mathbb{S}^2 \tag{5.6}$$

representing Hopf's class. Although the construction as such has already been given in [8], we here identify the domain poset as $(\mathbb{S}^3)'$ and define a specific order-preserving function in terms of the notation developed in Section 2. We modify table 1.1 so as to read

	j	$j+k$	k	$-j+k$	$-j$	$-j-k$	$-k$	$j-k$
1	i	j	$-i$	$-i$	$-i$	$-j$	i	i
$1+i$	j	j	$-i$	$-j$	$-j$	$-j$	i	j
i	$-i$	$-i$	$-i$	$-j$	i	i	i	j
$-1+i$	$-i$	$-j$	$-j$	$-j$	i	j	j	j
-1	$-i$	$-j$	i	i	i	j	$-i$	$-i$
$-1-i$	$-j$	$-j$	i	j	j	j	$-i$	$-j$
$-i$	i	i	i	j	$-i$	$-i$	$-i$	$-j$
$1-i$	i	j	j	j	$-i$	$-j$	$-j$	$-j$

(5.6a)

The modified table determines the assignment for elements of $(\mathbb{S}^3)'$ of the form

$$\varepsilon_1 1 + \varepsilon_2 i + \varepsilon_3 j + \varepsilon_4 k \quad (\varepsilon_1^2 + \varepsilon_2^2 \neq 0, \;\; \varepsilon_3^2 + \varepsilon_4^2 \neq 0).$$

The remaining elements are sent to $\hat{n}=1$ (if $\varepsilon_3=\varepsilon_4=0$) or to $\hat{s}=-1$ (if $\varepsilon_1=\varepsilon_2=0$). We may check that the assignment yields an order-preserving function

$$\Gamma(m) : (\mathbb{S}^3)' \to \mathbb{S}^2. \tag{5.6b}$$

5.1. Final remarks

Using the map $v : \mathrm{op}(\mathbb{S}^3)' \times (\mathbb{S}^3)' \to \mathbb{S}^3$ obtained in Theorem 2.1 we may consider the Hopf construction

$$\Gamma(v) : \mathrm{op}(\mathbb{S}^3)' \circledast (\mathbb{S}^3)' \to \mathbb{S}\mathbb{S}^3. \tag{5.7}$$

Certainly, the domain of this map is a finite model of the seven-dimensional sphere but if we consider the class of $|\mathscr{K}\Gamma(v)|$ in $\pi_7(S^4)$, it is not clear that it coincides with Hopf's class v. However, from the properties of v given in Theorem 2.1, it follows that $|\mathscr{K}v|$ is a map of 'type (1,1)' and hence the class obtained has Hopf–James invariant one. For details, see [15].

Appendix A. QBASIC program

```
REM The program checks order-preservation of v, a function from the product of the
   dual space
REM of (S^3)' with (S^3)' itself into S^3. To run it you will have to read first the
   comment below
REM (after the PRINT "STAR" command) and activate one of the options. When run, the
   program searches
REM for a failure of continuity at an entry in the t'th row (or the u'th column) of
   the table of values.
REM (You have to supply a value of t or a value of u as the case may be.) It prints
   out the numbers
REM at which no failure occurs. Otherwise the problem is printed out.
REM The program has long lines near the end. Continuations of lines are indented.
DIM n(40), p(40), q(40), R(40), s(40), m(9,9), v(40,40), ni(3000), D1(40), D2(40),
   D3(40),
DIM D4(40), D5(40), vv(40,40), ab(40,40), Di(40), Dj(40)
DEF FNF (n) = INT((n + 1)/2)
DEF FNLE (a,b) = FNF(a) < FNF(b) OR a = b
REM The points in L are enumerated. Convention: 1=1, 2=-1, 3=i, 4=-i, 5=j, 6=-j,
   7=k, 8=-k.
n(1) = 1000: n(2) = 300: n(3) = 50: n(4) = 7: n(5) = 1300: n(6) = 1050
n(7) = 1007: n(8) = 1400: n(9) = 1060: n(10) = 1008: n(11) = 350: n(12) = 307
n(13) = 57: n(14) = 360: n(15) = 308: n(16) = 58: n(17) = 357: n(18) = 367
n(19) = 358: n(20) = 368: n(21) = 1350: n(22) = 1307: n(23) = 1057
n(24) = 1360: n(25) = 1308: n(26) = 1058: n(27) = 1450: n(28) = 1407
n(29) = 1067: n(30) = 1460: n(31) = 1408: n(32) = 1068: n(33) = 1357
n(34) = 1457: n(35) = 1367: n(36) = 1358: n(37) = 1467: n(38) = 1458
n(39) = 1368: n(40) = 1468
ni(1000) = 1: ni(300) = 2: ni(50) = 3: ni(7) = 4: ni(1300) = 5
ni(1050) = 6: ni(1007) = 7: ni(1400) = 8: ni(1060) = 9: ni(1008) = 10
ni(350) = 11: ni(307) = 12: ni(57) = 13: ni(360) = 14: ni(308) = 15
```

```
ni(58) = 16: ni(357) = 17: ni(367) = 18: ni(358) = 19: ni(368) = 20
ni(1350) = 21: ni(1307) = 22: ni(1057) = 23: ni(1360) = 24: ni(1308) = 25
ni(1058) = 26: ni(1450) = 27: ni(1407) = 28: ni(1067) = 29: ni(1460) = 30
ni(1408) = 31: ni(1068) = 32: ni(1357) = 33: ni(1457) = 34: ni(1367) = 35
ni(1358) = 36: ni(1467) = 37: ni(1458) = 38: ni(1368) = 39: ni(1468) = 40
FOR m = 1 TO 40
p(m) = INT(n(m)/1000)
q(m) = INT((n(m) - 1000 * p(m))/100)
R(m) = INT((n(m) - 1000 * p(m) - 100 * q(m))/10)
s(m) = (n(m) - 1000 * p(m) - 100 * q(m) - 10 * R(m))
REM The next 3 lines give an opportunity to check that the numbers have
REM been entered correctly and the computation of p(m)=coefficient of 1,
REM q(m) coefficient of i, etc. (Activate by removing two "REM".)
REM PRINT p(m); q(m); r(m); s(m),
NEXT m
REM STOP
REM The assignment of values to the last 40 vertices is extended by "conjugation".
conj(1) = 2: conj(2) = 1: conj(3) = 4: conj(4) = 3: conj(5) = 6: conj(6) = 5: conj(7)
   = 8: conj(8) = 7
REM The following steps define the D operators
DEF FND1 (n) = ni(1000 * p(n) + 100 * q(n) + 10 * R(n))
DEF FND2 (n) = ni(1000 * p(n) + 100 * q(n) + s(n))
DEF FND3 (n) = ni(1000 * p(n) + 10 * R(n) + s(n))
DEF FND4 (n) = ni(100 * q(n) + 10 * R(n) + s(n))
DEF FNDD4 (n) = ni(100 * conj(q(n)) + 10 * conj(R(n)) + conj(s(n)))
DEF FNDD3 (n) = ni(1000 * conj(p(n)) + 10 * conj(R(n)) + conj(s(n)))
DEF FNDD2 (n) = ni(1000 * conj(p(n)) + 100 * conj(q(n)) + conj(s(n)))
DEF FNv (u,t) = v(ABS(u), ABS(t))
FOR n = 1 TO 40
D1(n) = FND1(n): IF D1(n) = n THEN D1(n) = 0
D2(n) = FND2(n): IF D2(n) = n THEN D2(n) = 0
D3(n) = FND3(n): IF D3(n) = n THEN D3(n) = 0
D4(n) = FND4(n)
IF FND2(n) = 0 THEN D2(n) = -FNDD2(n): IF FND3(n) = 0 THEN D3(n) = -FNDD3(n)
IF FND4(n) = 0 THEN D4(n) = -FNDD4(n): IF D4(n) = n THEN D4(n) = 0
REM The following enables the D operators to be checked.
REM PRINT p(n); q(n); r(n); s(n); "---"; (D1(n)); (D2(n)); (D3(n)); (D4(n))
NEXT n
REM Stage (2) We now read in the assignment.
v(1, 1) = 1: v(1, 2) = 3: v(1, 3) = 5: v(1, 4) = 7: v(1, 5) = 3
v(1, 6) = 5: v(1, 7) = 7: v(1, 8) = 4: v(1, 9) = 6: v(1, 10) = 8
v(1, 11) = 5: v(1, 12) = 7: v(1, 13) = 7: v(1, 14) = 6: v(1, 15) = 8
v(1, 16) = 8: v(1, 17) = 7: v(1, 18) = 7: v(1, 19) = 8: v(1, 20) = 8
v(1, 21) = 5: v(1, 22) = 7: v(1, 23) = 7: v(1, 24) = 6: v(1, 25) = 8
v(1, 26) = 8: v(1, 27) = 5: v(1, 28) = 7: v(1, 29) = 7: v(1, 30) = 6
```

```
v(1, 31) = 8: v(1, 32) = 8: v(1, 33) = 7: v(1, 34) = 7: v(1, 35) = 7
v(1, 36) = 8: v(1, 37) = 7: v(1, 38) = 8: v(1, 39) = 8: v(1, 40) = 8
REM The remainder of the assignment may be obtained from tables 1 to 4.
PRINT: PRINT "STAR"
REM The following steps enable continuity checks for rows respectively
REM columns. To check a row, choose a value for t and activate line 5 by
REM removing the REM comment just below. For columns, activate line 7.
REM t = 32: GOTO 5
REM u = 2: GOTO 7
5 PRINT "ROW t = "; t
FOR u = 5 TO 40
v = v(t, u)
iatu = v(t, ABS(D1(u))): IF D1(u) < 0 THEN iatu = conj(iatu)
ibtu = v(t, ABS(D2(u))): IF D2(u) < 0 THEN ibtu = conj(ibtu)
ictu = v(t, ABS(D3(u))): IF D3(u) < 0 THEN ictu = conj(ictu)
idtu = v(t, ABS(D4(u))): IF D4(u) < 0 THEN idtu = conj(idtu)
satu = v(ABS(D1(t)),u): IF D1(t) < 0 THEN satu = conj(satu)
sbtu = v(ABS(D2(t)),u): IF D2(t) < 0 THEN sbtu = conj(sbtu)
sctu = v(ABS(D3(t)),u): IF D3(t) < 0 THEN sctu = conj(sctu)
sdtu = v(ABS(D4(t)),u): IF D4(t) < 0 THEN sdtu = conj(sdtu)
IF iatu = 9 THEN iatu = 0: IF ibtu = 9 THEN ibtu = 0
IF ictu = 9 THEN ictu = 0: IF idtu = 9 THEN idtu = 0
IF satu = 0 THEN satu = 9: IF sbtu = 0 THEN sbtu = 9
IF sctu = 0 THEN sctu = 9: IF sdtu = 0 THEN sdtu = 9
IF FNLE(iatu,v) AND FNLE(ibtu,v) AND FNLE(ictu,v) AND FNLE(idtu,v) AND
   FNLE(v,satu)
AND FNLE(v,sbtu) AND FNLE(v,sctu) AND FNLE(v,sdtu) THEN PRINT u; ELSE GOTO 10
NEXT u
GOTO 12
7 PRINT "Column u = "; u
FOR t = 5 TO 40
8 v = v(t, u)
iatu = v(t, ABS(D1(u))): IF D1(u) < 0 THEN iatu = conj(iatu)
ibtu = v(t, ABS(D2(u))): IF D2(u) < 0 THEN ibtu = conj(ibtu)
ictu = v(t, ABS(D3(u))): IF D3(u) < 0 THEN ictu = conj(ictu)
idtu = v(t, ABS(D4(u))): IF D4(u) < 0 THEN idtu = conj(idtu)
satu = v(ABS(D1(t)), u): IF D1(t) < 0 THEN satu = conj(satu)
sbtu = v(ABS(D2(t)), u): IF D2(t) < 0 THEN sbtu = conj(sbtu)
sctu = v(ABS(D3(t)), u): IF D3(t) < 0 THEN sctu = conj(sctu)
sdtu = v(ABS(D4(t)), u): IF D4(t) < 0 THEN sdtu = conj(sdtu)
IF iatu = 9 THEN iatu = 0: IF ibtu = 9 THEN ibtu = 0
IF ictu = 9 THEN ictu = 0: IF idtu = 9 THEN idtu = 0
IF satu = 0 THEN satu = 9: IF sbtu = 0 THEN sbtu = 9
IF sctu = 0 THEN sctu = 9: IF sdtu = 0 THEN sdtu = 9
IF FNLE(iatu, v) AND FNLE(ibtu, v) AND FNLE(ictu, v) AND FNLE(idtu, v) AND
```

```
   FNLE(v, satu)
AND FNLE(v, sbtu) AND FNLE(v, sctu) AND FNLE(v, sdtu) THEN PRINT t; ELSE GOTO 11
NEXT t
GOTO 12
10 PRINT : IF u <= 40 THEN PRINT "FAIL "; t; " x "; u; " = "; v, (iatu); (ibtu);
   (ictu); (idtu); " --- ";
    (satu); (sbtu); (sctu); (sdtu)
PRINT D1(u); D2(u); D3(u); D4(u)
GOTO 12
11 PRINT: IF t <= 40 THEN PRINT "FAIL "; t; " x "; u; " = "; v, (iatu); (ibtu);
   (ictu); (idtu); " --- ";
    (satu); (sbtu); (sctu); (sdtu) PRINT D1(t); D2(t); D3(t); D4(t)
12
```

References

[1] P.S. Alexandroff, Diskrete Räume, Matematiceskii Sbornik (N.S.) 2 (1937) 501–518.

[2] A. Björner, Topological methods, in: R. Graham, M. Grötschel, L. Lewis (Eds.), Handbook of Combinatorics, North-Holland, Amsterdam, 1995.

[3] J.M. Boardman, B. Steer, On Hopf invariants, Comment. Math. Helv. 42 (1967) 180–221.

[4] J. Dieudonné, A History of Algebraic and Differential Topology 1900–1960, Birkhäuser, Basel, 1989.

[5] H. Hopf, Über die Abbildungen der dreidimensionalen Sphäre auf die Kugelfläche, Math. Ann. 104 (1931) 637–665.

[6] H. Hopf, Über die Abbildungen von Sphären auf Sphären von niedriger Dimension, Fund. Math. 25 (1935) 427–440.

[7] K.A. Hardie, J.J.C. Vermeulen, Homotopy theory of finite and locally finite T_0-spaces, Expo. Math. 11 (1993) 331–341.

[8] K.A. Hardie, J.J.C. Vermeulen, P.J. Witbooi, A non-trivial pairing of finite T_0-spaces, Topol. Appl., to appear.

[9] P.J. Hilton, S. Wylie, Homology Theory, Cambridge University Press, Cambridge, 1960.

[10] M.C. McCord, Singular homology groups and homotopy groups of finite topological spaces, Duke Math. J. 33 (1966) 465–474.

[11] M. Nakahara, Geometry Topology and Physics, Institute of Physics Publishing, Bristol and Philadelphia, 1990.

[12] N. Oda, The homotopy set of the axes of pairings, Canad. J. Math. 42 (1990) 856–868.

[13] D. Quillen, Higher Algebraic K-Theory: I, in: Lecture Notes in Mathematics, Vol. 341, Springer, Berlin, Heidelberg, New York, 1973.

[14] R.E. Stong, Finite topological spaces, Trans. Amer. Math. Soc. 123 (1966) 325–340.

[15] H. Toda, Composition Methods in Homotopy Groups of Spheres, in: Annals of Mathematics Studies, Vol. 49, Princeton University Press, Princeton, 1962.

[16] G.W. Whitehead, Elements of Homotopy Theory, Springer, New York, Heidelberg, Berlin, 1978.

[17] Peter Witbooi, Finite spaces and the universal bundle of a group, Comment. Math. Univ. Carolinae 38 (1997) 791–799.